CS 61A Spring 2022 — Interpreters, Tail Calls

Discussion 11: April 6, 2022 Solutions

Tail Recursion

When writing a recursive procedure, it's possible to write it in a **tail recursive** way, where all of the recursive calls are tail calls. A **tail call** occurs when a function calls another function as the last action of the current frame.

Consider this implementation of `factorial` that is *not* tail recursive:

```
(define (factorial n)
  (if (= n 0)
      1
      (* n (factorial (- n 1)))))
```

The recursive call occurs in the last line, but it is not the last expression evaluated. After calling `(factorial (- n 1))`, the function still needs to multiply that result with `n`. The final expression that is evaluated is a call to the multiplication function, not `factorial` itself. Therefore, the recursive call is `not` a tail call.

Here's a visualization of the recursive process for computing `(factorial 6)` :

```
(factorial 6)
(* 6 (factorial 5))
(* 6 (* 5 (factorial 4)))
(* 6 (* 5 (* 4 (factorial 3))))
(* 6 (* 5 (* 4 (* 3 (factorial 2)))))
(* 6 (* 5 (* 4 (* 3 (* 2 (factorial 1))))))
(* 6 (* 5 (* 4 (* 3 (* 2 1)))))
(* 6 (* 5 (* 4 (* 3 2))))
(* 6 (* 5 (* 4 6)))
(* 6 (* 5 24))
(* 6 120)
720
```

The interpreter first must reach the base case and only then can it begin to calculate the products in each of the earlier frames.

We can rewrite this function using a helper function that remembers the temporary product that we have calculated so far in each recursive step.

```
(define (factorial n)
  (define (fact-tail n result)
    (if (= n 0)
        result
        (fact-tail (- n 1) (* n result))))
  (fact-tail n 1))
```

`fact-tail` makes a single recursive call to `fact-tail`, and that recursive call is the last expression to be evaluated, so it is a tail call. Therefore, `fact-tail` is a tail recursive process.

Here's a visualization of the tail recursive process for computing `(factorial 6)`:

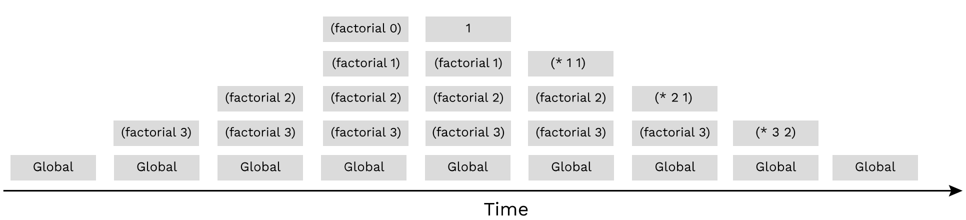
```
(factorial 6)
(fact-tail 6 1)
(fact-tail 5 6)
(fact-tail 4 30)
(fact-tail 3 120)
(fact-tail 2 360)
(fact-tail 1 720)
(fact-tail 0 720)
720
```

The interpreter needed less steps to come up with the result, and it didn't need to re-visit the earlier frames to come up with the final product.

Tail Call Optimization

When a recursive procedure is not written in a tail recursive way, the interpreter must have enough memory to store all of the previous recursive calls.

For example, a call to the `(factorial 3)` in the non tail-recursive version must keep the frames for all the numbers from 3 down to the base case, until it's finally able to calculate the intermediate products and forget those frames:



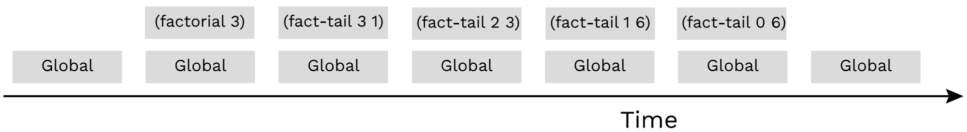
Example Tree

For non tail-recursive procedures, the number of active frames grows proportionally to the number of recursive calls. That may be fine for small inputs, but imagine calling `factorial` on a large number like 10000. The interpreter would need enough memory for all 1000 calls!

Note: This worksheet is a problem bank—most TAs will not cover all the problems in discussion section.

Fortunately, proper Scheme interpreters implement **tail-call optimization** as a requirement of the language specification. TCO ensures that tail recursive procedures can execute with a constant number of active frames, so programmers can call them on large inputs without fear of exceeding the available memory.

When the tail recursive `factorial` is run in an interpreter with tail-call optimization, the interpreter knows that it does not need to keep the previous frames around, so it never needs to store the whole stack of frames in memory:



Example Tree

Tail-call optimization can be implemented in a few ways:

1. Instead of creating a new frame, the interpreter can just update the values of the relevant variables in the current frame (like `n` and `result` for the `fact-tail` procedure). It reuses the same frame for the entire calculation, constantly changing the bindings to match the next set of parameters.
2. How our 61A Scheme interpreter works: The interpreter builds a new frame as usual, but then *replaces* the current frame with the new one. The old frame is still around, but the interpreter no longer has any way to get to it. When that happens, the Python interpreter does something clever: it *recycles* the old frame so that the next time a new frame is needed, the system simply allocates it out of recycled space. The technical term is that the old frame becomes "garbage", which the system "garbage collects" behind the programmer's back.

Tail Context

When trying to identify whether a given function call within the body of a function is a tail call, we look for whether the call expression is in **tail context**.

Given that each of the following expressions is the last expression in the body of the function, the following expressions are tail contexts:

1. the second or third operand in an `if` expression
2. any of the non-predicate sub-expressions in a `cond` expression (i.e. the second expression of each clause)
3. the last operand in an `and` or an `or` expression
4. the last operand in a `begin` expression's body
5. the last operand in a `let` expression's body

For example, in the expression `(begin (+ 2 3) (- 2 3) (* 2 3))`, `(* 2 3)` is a tail call because it is the last operand expression to be evaluated.

Tail calls

Q1: Is Tail Call

For each of the following procedures, identify whether it contains a recursive call in a tail context. Also indicate if it uses a constant number of active frames.

```
(define (question-a x)
  (if (= x 0) 0
      (+ x (question-a (- x 1)))))
```

In the recursive case, the last expression that is evaluated is a call to `+`. Therefore, the recursive call is not in tail context, and each of the frames remain active. This procedure uses a number of active frames proportional to the input `x`.

```
(define (question-b x y)
  (if (= x 0) y
      (question-b (- x 1) (+ y x))))
```

The recursive call is the third operand in the if expression, so it is in tail context. This means that the last expression that will be evaluated in the body of this procedure is the recursive procedure call, so this procedure can be run with a constant number of active frames.

```
(define (question-c x y)
  (if (> x y)
      (question-c (- y 1) x)
      (question-c (+ x 10) y)))
```

The recursive calls are the second and third operands of the `if` expression. Only one of these calls is actually evaluated, and whichever one it is will be the last expression evaluated in the body of the procedure. This procedure therefore can be run with a constant number of active frames.

Note that if you actually try and evaluate this procedure, it will never terminate. But at least it won't crash from hitting max recursion depth!

```
(define (question-d n)
  (if (question-d n)
      (question-d (- n 1))
      (question-d (+ n 10))))
```

The second and third recursive calls are in tail context, but the first is not. Since not all the recursive calls are tail calls, this procedure requires active frames for all of the recursive calls.

Additionally, this question will actually lead to infinite recursion because the if condition will never reach a base case!

```
(define (question-e n)
  (cond ((<= n 1) 1)
        ((question-e (- n 1)) (question-e (- n 2)))
        (else (begin (print 2) (question-e (- n 3))))))
```

The second and third recursive calls are the second expressions in a clause, so they are in tail context. However, the first recursive call is not in tail context. Since not all recursive calls are tail calls, this procedure is not tail recursive and does not use a constant number of active frames.

Q2: Sum

Write a tail recursive function that takes in a Scheme list and returns the numerical sum of all values in the list. You can assume that the list contains only numbers (no nested lists).

```
scm> (sum '(1 2 3))
6
scm> (sum '(10 -3 4))
11
```

```
(define (sum lst)
  (define (sum-sofar lst current-sum)
    (if (null? lst)
        current-sum
        (sum-sofar (cdr lst) (+ (car lst) current-sum))))
  (sum-sofar lst 0)

)

; ALTERNATE SOLUTION
(define (sum lst)
    (cond
      ((null? lst) 0)
      ((null? (cdr lst)) (car lst))
      (else (sum (cons (+ (car lst) (car (cdr lst))) (cdr (cdr lst))
    )))
    )
)

(expect (sum '(1 2 3)) 6)
(expect (sum '(10 -3 4)) 11)
```

Video walkthrough

Q3: Reverse

Write a tail-recursive function `reverse` that takes in a Scheme list a returns a reversed copy. *Hint*: use a helper function!

```
scm> (reverse '(1 2 3))
(3 2 1)
scm> (reverse '(0 9 1 2))
(2 1 9 0)
```

```
(define (reverse lst)
  (define (reverse-tail sofar rest)
    (if (null? rest)
          sofar
          (reverse-tail (cons (car rest) sofar) (cdr rest))))
  (reverse-tail nil lst)
)

(expect (reverse '(1 2 3)) (3 2 1))
(expect (reverse '(0 9 1 2)) (2 1 9 0))
```

Calculator

An interpreter is a program that understands other programs. Today, we will explore how to build an interpreter for Calculator, a simple language that uses a subset of Scheme syntax.

The Calculator language includes only the four basic arithmetic operations: `+`, `-`, `*`, and `/`. These operations can be nested and can take any numbers of arguments. A few examples of calculator expressions and their corresponding values are shown below.

```
calc> (+ 2 2)
4

calc> (- 5)
-5

calc> (* (+ 1 2) (+ 2 3))
15
```

The reader component of an interpreter parses input strings and represents them as data structures in the implementing language. In this case, we need to represent Calculator expressions as Python objects. To represent numbers, we can just use

Python numbers. To represent the names of the arithmetic procedures, we can use Python strings (e.g. `'+'`).

To represent Scheme lists in Python, we will use the `Pair` class. A `Pair` instance holds exactly two elements. Accordingly, the `Pair` constructor takes in two arguments, and to make a list we must nest calls to the constructor and pass in `nil` as the second element of the last pair. Note that in the Python code, `nil` is bound to a special user-defined object that represents an empty list, whereas `nil` in Scheme is actually an empty list.

```
>>> Pair('+', Pair(2, Pair(3, nil)))
Pair('+', Pair(2, Pair(3, nil)))
```

Each `Pair` instance has two instance attributes: `first` and `rest`, which are bound to the first and second elements of the pair respectively.

```
>>> p = Pair('+', Pair(2, Pair(3, nil)))
>>> p.first
'+'
>>> p.rest
Pair(2, Pair(3, nil))
>>> p.rest.first
2
```

`Pair` is very similar to `Link`, the class we developed for representing linked lists – they have the same attribute names `first` and `rest` and are represented very similarly. Here's an implementation of what we described:

```
class Pair:
    """Represents the built-in pair data structure in Scheme."""
    def __init__(self, first, rest):
        self.first = first
        if not scheme_valid_cdrp(rest):
            raise SchemeError("cdr can only be a pair, nil, or a
    promise but was {}".format(rest))
        self.rest = rest

    def map(self, fn):
        """Maps fn to every element in a list, returning a new
        Pair.

        >>> Pair(1, Pair(2, Pair(3, nil))).map(lambda x: x * x)
        Pair(1, Pair(4, Pair(9, nil)))
        """
        assert isinstance(self.rest, Pair) or self.rest is nil, \
            "rest element in pair must be another pair or nil"
        return Pair(fn(self.first), self.rest.map(fn))

    def __repr__(self):
        return 'Pair({}, {})'.format(self.first, self.rest)
```

```
class nil:
    """Represents the special empty pair nil in Scheme."""
    def map(self, fn):
        return nil
    def __getitem__(self, i):
         raise IndexError('Index out of range')
    def __repr__(self):
        return 'nil'

nil = nil() # this hides the nil class *forever*
```

Q4: Using Pair

Answer the following questions about a `Pair` instance representing the Calculator expression `(+ (- 2 4) 6 8)`.

Write out the Python expression that returns a `Pair` representing the given expression:

```
>>> Pair('+', Pair(Pair('-', Pair(2, Pair(4, nil))), Pair(6, Pair(8,
    nil))))
```

What is the operator of the call expression?

-

If the `Pair` you constructed in the previous part was bound to the name `p`, how would you retrieve the operator?

`p.first`

What are the operands of the call expression?

An expression (- 2 4), the number 6, the number 8.

If the `Pair` you constructed was bound to the name `p`, how would you retrieve a list containing all of the operands?

`p.rest`

How would you retrieve only the first operand?

`p.rest.first`

Q5: New Procedure

Suppose we want to add the `//` operation to our Calculator interpreter. Recall from Python that `//` is the floor division operation, so we are looking to add a built-in procedure `//` in our interpreter such that `(// dividend divisor)` returns dividend // divisor. Similarly we handle multiple inputs as illustrated in the following example `(// dividend divisor1 divisor2 divisor3)` evaluates to (((dividend // divisor1) // divisor2) // divisor3). For this problem you can assume you are always given at least 1 divisor. Also for this question do you need to call `calc_eval` inside `floor_div`? Why or why not?

```
calc> (// 1 1)
1
calc> (// 5 2)
2
calc> (// 28 (+ 1 1) 1)
14
```

```
def calc_eval(exp):
    if isinstance(exp, Pair): # Call expressions
        return calc_apply(calc_eval(exp.first), exp.rest.map(
    calc_eval))
    elif exp in OPERATORS:      # Names
        return OPERATORS[exp]
    else:                       # Numbers
        return exp

def floor_div(expr):
    """
    >>> calc_eval(Pair("//", Pair(10, Pair(10, nil))))
    1
    >>> calc_eval(Pair("//", Pair(20, Pair(2, Pair(5, nil)))))
    2
    >>> calc_eval(Pair("//", Pair(6, Pair(2, nil))))
    3
    """
    dividend = expr.first
    expr = expr.rest
    while expr != nil:
        divisor = expr.first
        dividend //= divisor
        expr = expr.rest
    return dividend

OPERATORS = { "//": floor_div }
```

Q6: New Form

Suppose we want to add handling for comparison operators `>`, `<`, and `=` as well as `and` expressions to our Calculator interpreter. These should work the same way they do in Scheme.

```
calc> (and (= 1 1) 3)
3
calc> (and (+ 1 0) (< 1 0) (/ 1 0))
#f
```

i. Are we able to handle expressions containing the comparison operators (such as `<`, `>`, or `=`) with the existing implementation of `calc_eval`? Why or why not?

Comparison expressions are regular call expressions, so we need to evaluate the operator and operands and then apply a function to the arguments. Therefore, we do not need to change `calc_eval`. We simply need to add new entries to the

OPERATORS dictionary that map '<', '>', and '=' to functions that perform the appropriate comparison operation.

ii. Are we able to handle and expressions with the existing implementation of calc_eval? Why or why not?

Hint: Think about the rules of evaluation we've implemented in calc_eval. Is anything different about and?

Since and is a special form that short circuits on the first false-y operand, we cannot handle these expressions the same way we handle call expressions. We need to add special handling for combinations that don't evaluate all the operands.

iii. Now, complete the implementation below to handle and expressions. You may assume the conditional operators (e.g. <, >, =, etc) have already been implemented for you.

```
def calc_eval(exp):
    if isinstance(exp, Pair):
        if exp.first == 'and': # and expressions
            return eval_and(exp.rest)
        else:                  # Call expressions
            return calc_apply(calc_eval(exp.first), exp.rest.map(
   calc_eval))
    elif exp in OPERATORS:     # Names
        return OPERATORS[exp]
    else:                      # Numbers
        return exp

def eval_and(operands):
    """
    >>> calc_eval(Pair("and", Pair(1, nil)))
    1
    >>> calc_eval(Pair("and", Pair(False, Pair("1", nil))))
    False
    """
    curr, val = operands, True
    while curr is not nil:
        val = calc_eval(curr.first)
        if val is False:
            return False
        curr = curr.rest
    return val

OPERATORS = {}
```

Q7: Saving Values

In the last few questions we went through a lot of effort to add operations so we can do most arithmetic operations easily. However it's a real shame we can't store these values. So for this question let's implement a `define` special form that saves values to variable names. This should work like variable assignment in Scheme; this means that you should expect inputs of the form`(define <variable_name> <value>)` and these inputs should return the symbol corresponding to the variable name.

```
calc> (define a 1)
a
calc> a
1
```

This is a more involved change. Here are the 4 steps involved: 1. Add a `bindings` dictionary that will store the names and correspondings values of variables as key-value pairs of the dictionary. 2. Identify when the define form is given to `calc_eval`. 3. Allow variables to be looked up in `calc_eval`. 4. Write the function `eval_define` which should actually handle adding names and values to the bindings dictionary.

We've done step 1 for you. Now you'll do the remaining steps in the code below.

```
bindings = {}
def calc_eval(exp):
    if isinstance(exp, Pair):
        if exp.first == 'and': # and expressions
            return eval_and(exp.rest)
        elif exp.first == 'define': # and expressions
            return eval_define(exp.rest)

        else:                  # Call expressions
            return calc_apply(calc_eval(exp.first), exp.rest.map(
    calc_eval))
    elif exp in bindings: # Looking up variables
        return bindings[exp]
    elif exp in OPERATORS:      # Looking up procedures
        return OPERATORS[exp]
    else:                       # Numbers
        return exp

def eval_define(expr):
    """
    >>> calc_eval(Pair("define", Pair("a", Pair(1, nil))))
    'a'
    >>> calc_eval("a")
    1
    """
    name, value = expr.first, calc_eval(expr.rest.first)
    bindings[name] = value
    return name

OPERATORS = {}
```

Q8: Counting Eval and Apply

How many calls to `calc_eval` and `calc_apply` would it take to evaluate each of the following Calculator expressions?

`scm> (+ 1 2)`

For this particular prompt please list out the inputs to `calc_eval` and `calc_apply`.

4 calls to eval: 1 for the entire expression, and then 1 each for the operator and each operand.

1 call to apply the addition operator.

Explicity listing out the inputs we have the following for calc_eval: , '+', 1, 2. calc_apply is given '+' for fn and (1 2) for args.

A note is that (+ 1 2) corresponds to the following Pair, Pair('+', Pair(1, Pair(2, nil))) and (1 2) corresponds to the Pair, Pair(1, Pair(2, nil)).

```
scm> (+ 2 4 6 8)
```

6 calls to eval: 1 for the entire expression, and then 1 each for the operator and each operand.

1 call to apply the addition operator.

```
scm> (+ 2 (* 4 (- 6 8)))
```

10 calls to eval: 1 for the whole expression, then 1 for each of the operators and operands. When we encounter another call expression, we have to evaluate the operators and operands inside as well.

3 calls to apply the function to the arguments for each call expression.

```
scm> (and 1 (+ 1 0) 0)
```

7 calls to eval: 1 for the whole expression, 1 for the first argument, 1 for `(+ 1 0)`, 1 for the `+` operator, 2 for the operands to plus, and 1 for the final 0. Notice that and is a special form so we do not run `calc_eval` on the and.

1 calls to apply to evaluate the `+` expression.

Video Walkthrough

Q9: From Pair to Calculator

Write out the Calculator expression with proper syntax that corresponds to the following `Pair` constructor calls.

```
>>> Pair('+', Pair(1, Pair(2, Pair(3, Pair(4, nil)))))
```

```
> (+ 1 2 3 4)
```

```
>>> Pair('+', Pair(1, Pair(Pair('*', Pair(2, Pair(3, nil))), nil)))
```

```
> (+ 1 (* 2 3))
```

Box and pointers solutions Video walkthrough