Recursion

Class outline:

- Recursive functions
- Recursion in environment diagrams
- Mutual recursion
- Recursion vs. iteration

Recursive functions

A function is **recursive** if the body of that function calls itself, either directly or indirectly.

Recursive functions often operate on increasingly smaller instances of a problem.

Circle Limit, by M.C. Escher

Summing digits

$2 + 0 + 2 + 1 = 5$

Summing digits

2 + 0 + 2 + 1 = 5

Fun fact: The sum of the digits in a multiple of 9 is also divisible by 9.

9 * 82 = 738

7 + 3 + 8 = 18

The problems within the problem

The sum of the digits of 6 is simply 6.
Generally: the sum of any one-digit non-negative number is that number.

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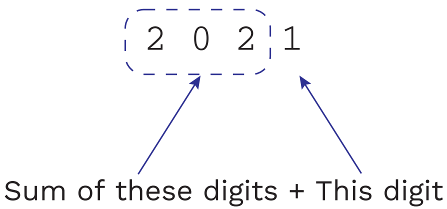
The sum of the digits of 2021 is:



Generally: the sum of a number is the sum of the first digits (`number // 10`), plus the last digit (`number % 10`).

Summing digits without a loop

```
def sum_digits(n):
    """Return the sum of the digits of positive integer n.
    >>> sum_digits(6)
    6
    >>> sum_digits(2021)
    5
    """
```

Summing digits without a loop

```
def sum_digits(n):
    """Return the sum of the digits of positive integer n.
    >>> sum_digits(6)
    6
    >>> sum_digits(2021)
    5
    """
    if n < 10:
        return n
    else:
        all_but_last = n // 10
        last = n % 10
        return sum_digits(all_but_last) + last
```

Anatomy of a recursive function

- **Base case**: Evaluated without a recursive call (the smallest subproblem).
- **Recursive case**: Evaluated with a recursive call (breaking down the problem further)
- **Conditional statement** to decide if it's a base case

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def sum_digits(n):
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```
def sum_digits(n):
    if n < 10: # BASE CASE
        return n
    else:          # RECURSIVE CASE
        all_but_last = n // 10
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    return sum_digits(all_but_last) + last
```

Visualizing recursion

Recursive factorial

The factorial of a number is defined as:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$$

```
def fact(n):
    """
    >>> fact(0)
    1
    >>> fact(4)
    24
    """
```

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```
def fact(n):
    """
    >>> fact(0)
    1
    >>> fact(4)
    24
    """
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```

Recursive call visualization

```
def fact(n):
    if n == 0:
        return 1
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fact(3)
```

```
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```

Recursive call visualization

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def fact(n):
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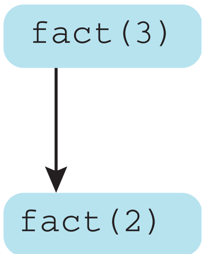
fact(3)
fact(2)

Recursive call visualization

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        return 1
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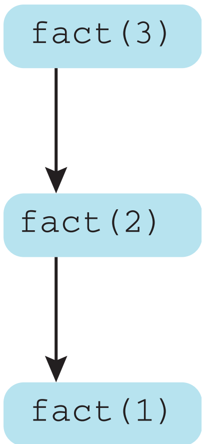
fact(3)
```


fact(3)
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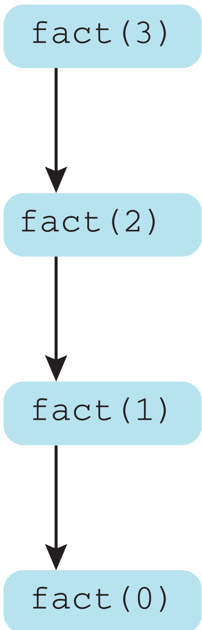
fact(3)
fact(2)
fact(1)
fact(0)

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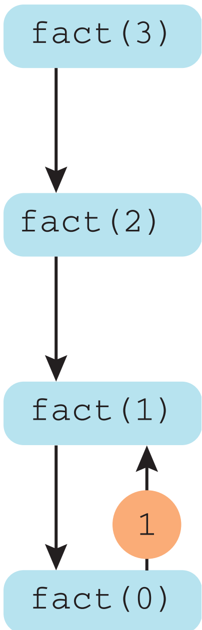
fact(3)
fact(2)
fact(1)
1
fact(0)

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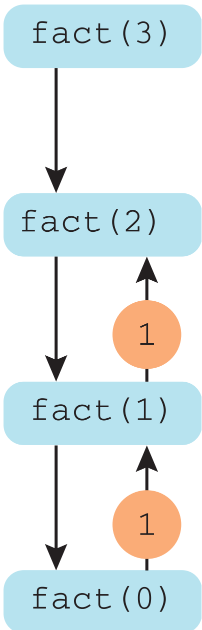
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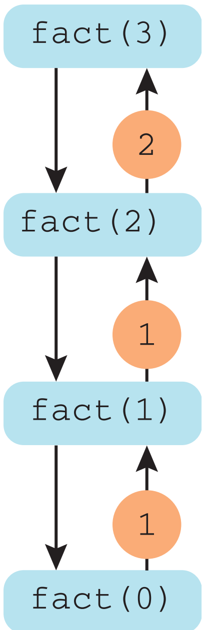
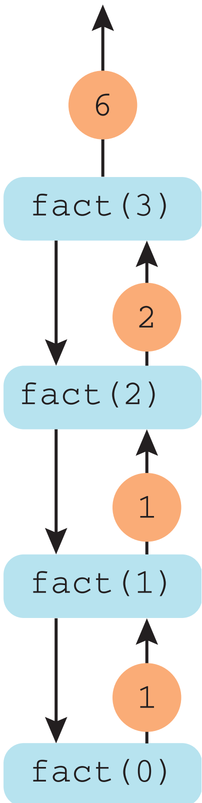

fact(3)
2
fact(2)
1
fact(1)
1
fact(0)

Recursive call visualization

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)

fact(3)
```


6
fact(3)
2
fact(2)
1
fact(1)
1
fact(0)

Recursion in environment diagrams

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)

fact(3)
```

- The same function `fact` is called multiple times
- Different frames keep track of the different arguments in each call
- What `n` evaluates to depends upon the current environment
- Each call to `fact` solves a simpler problem than the last: smaller `n`

```
Global frame
        fact │→ func fact[parent=Global]

f1:

      Return │
```

value

f2:

Return
value

f3:

Return
value

f4:

Recursion in environment diagrams

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def fact(n):
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fact(3)
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```
Global frame
        fact │→ func fact[parent=Global]

f1: fact[parent=Global]
           n │3
      Return │6
```

value

f2: fact[parent=Global]

n	2
Return value	2

f3: fact[parent=Global]

n	1
Return value	1

f4: fact[parent=Global]

n	0
Return value	1

Verifying recursive functions

Falling dominoes

If a million dominoes are equally spaced out and we tip the first one, will they all fall?

1. Verify that one domino will fall, if tipped
2. Assume that any given domino falling over will tip the next one over
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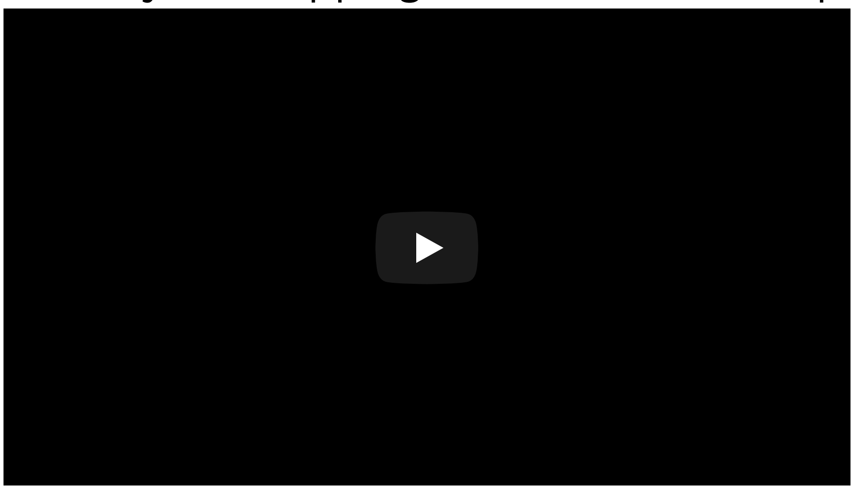

The recursive leap of faith

```
def fact(n):
    """Returns the factorial of N."""
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```

Is `fact` implemented correctly?

1. Verify the base case
2. Treat `fact` as a functional abstraction!
3. Assume that `fact(n-1)` is correct (← the leap!)
4. Verify that `fact(n)` is correct

The recursive elf's promise

Imagine we're trying to compute 5!

We ask ourselves, "If I somehow knew how to compute 4!, could I compute 5!?"

Credit: FuschiaKnight, r/compsci

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Yep, 5! = 5 * 4!

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The recursive elf's promise

Imagine we're trying to compute 5!

We ask ourselves, "If I somehow knew how to compute 4!, could I compute 5!?"

Yep, 5! = 5 * 4!

♀ The `fact()` function promises, "hey friend, tell you what, while you're working hard on 5!, I'll compute 4! for you, and you can finish it off!"

Credit: FuschiaKnight, r/compsci

Mutual recursion

The Luhn algorithm

Used to verify that a credit card numbers is valid.

1. From the rightmost digit, which is the check digit, moving left, double the value of every second digit; if product of this doubling operation is greater than 9 (e.g., 7 * 2 = 14), then sum the digits of that product (e.g., 14: 1 + 4 = 5)
2. Take the sum of all the digits

Original	1	3	8	7	4	3
Processed						

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Processed			1+6=7	7	8	3

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Original	1	3	8	7	4	3
Processed		3	1+6=7	7	8	3

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Original	1	3	8	7	4	3
Processed	2	3	1+6=7	7	8	3

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Original	1	3	8	7	4	3	
Processed	2	3	1+6=7	7	8	3	= 30

The Luhn sum of a valid credit card number is a multiple of 10

Calculating the Luhn sum

Let's start with...

```
def sum_digits(n):
    if n < 10:
        return n
    else:
        last = n % 10
        all_but_last = n // 10
        return last + sum_digits(all_but_last)

def luhn_sum(n):
    """Returns the Luhn sum for the positive number N.
    >>> luhn_sum(2)
    2
    >>> luhn_sum(32)
    8
    >>> luhn_sum(5105105105105100)
    20
    """
```

Luhn sum with mutual recursion

```
def luhn_sum(n):
    if n < 10:
        return n
    else:
        last = n % 10
        all_but_last = n // 10
        return last + luhn_sum_double(all_but_last)

def luhn_sum_double(n):
    last = n % 10
    all_but_last = n // 10
    luhn_digit = sum_digits(last * 2)
    if n < 10:
        return luhn_digit
    else:
        return luhn_digit + luhn_sum(all_but_last)
```

Recursion and Iteration

Recursion vs. iteration

	Using recursion:	**Using iteration:**

	`def fact(n):`
	`  if n == 0:`
	`    return 1`
	`  else:`
	`    return n * fact(n-1)`
		`def fact(n):`
		`  total = 1`
		`  k = 1`
		`  while k <= n:`
		`    total *= k`
		`    k += 1`
		`  return total`

Math:	$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$	$n! = \prod_{k=1}^{n} k$
Names:	`fact`, `n`	`fact`, `n`, `total`, `k`

Converting recursion to iteration

Can be tricky: Iteration is a special case of recursion.

Figure out what state must be maintained by the iterative function.

```
def sum_digits(n):
    if n < 10:
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```

```
def sum_digits(n):
    digit_sum = 0
    while n >= 10:
        last = n % 10
        n = n // 10
        digit_sum += last
    return digit_sum
```

Converting iteration to recursion

More formulaic: Iteration is a special case of recursion.

The state of an iteration can be passed as arguments.

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def sum_digits(n):
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The state of an iteration can be passed as arguments.

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def sum_digits(n):
    digit_sum = 0
    while n >= 10:
        last = n % 10
        n = n // 10
        digit_sum += last
    return digit_sum
```

```
def sum_digits(n, digit_sum):
    if n == 0:
        return digit_sum
    else:
        last = n % 10
        all_but_last = n // 10
        return sum_digits(all_but_last, digit_sum + last)
```